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OBSERVATIONS ON THE SEXTIC EQUATION WITH THREE UNKNOWNS $3(x^2 + y^2) - 2xy = 972z^6$

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ABSTRACT

This paper deals with the problem of finding non-zero distinct integer solutions to the non-homogeneous ternary sextic equation given by $3(x^2 + y^2) - 2xy = 972z^6$.

Key words: Non-homogeneous sextic, Ternary sextic, Integer solutions.

INTRODUCTION

It is well-known that a Diophantine equation is an algebraic equation with integer coefficients involving two or more unknowns such that the only solutions focused are integer solutions. No doubt that Diophantine equations are rich in variety [1-3, 18, 20, 21]. There is no universal method available to know whether a Diophantine equation has a solution or finding all solutions if it exists. For equations with more than three variables and degree at least three, very little is known. It seems that much work has not been done in solving higher degree Diophantine equations. While focusing the attention on solving Sextic Diophantine equations with variables at least three, the problems illustrated in [4-17, 19, 22, 23] are observed. This paper focuses on finding integer solutions to the sextic equation with three unknowns $3(x^2 + y^2) - 2xy = 972z^6$ and they differ from [23].

METHOD OF ANALYSIS

The non-homogeneous Diophantine equation of degree six with three unknowns to be solved in integers is

$$3(x^2 + y^2) - 2xy = 972z^6 \tag{1}$$

Introduction of the transformations

$$x = 72(u + v),\ y = 72(u - v),\ z = 2w \tag{2}$$

in (1) leads to

$$u^2 + 2v^2 = 3w^6 \tag{3}$$

Different ways of determining non-zero distinct integer solutions to (1) are illustrated below:

Way:

Assume

$$w = a^2 + 2b^2 \tag{4}$$

Write 3 on the R.HLS. of (3) as

$$3 = (1 + i\sqrt{2})(1 - i\sqrt{2}) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$u + i\sqrt{2}v = (1 + i\sqrt{2})(a - i\sqrt{2})^6$$

from which, on equating the real and imaginary parts, one obtains

$$u = a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6 - 2(6a^5b - 40a^3b^3 + 24ab^5) \tag{6}$$
$$v = a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6 + 6a^5b - 40a^3b^3 + 24ab^5$$

From (4), (6) and (2) one has

$$x = 72(2(a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6) - (6a^5b - 40a^3b^3 + 24ab^5)),$$
$$y = -216(6a^5b - 40a^3b^3 + 24ab^5), \tag{7}$$
$$z = 2a^2 + 4b^2$$

which represent the non-zero distinct integer solutions to (1).

Note 1:

One may also express 3 on the R.H.S. of (3) as below:

$$3 = \frac{(5+i\sqrt{2})(5-i\sqrt{2})}{9}$$

$$3 = \frac{(1+i11\sqrt{2})(1-i11\sqrt{2})}{25}$$

$$3 = \frac{(2k^2-3+i\sqrt{2})(2k^2+6k+3))(2k^2-3-i\sqrt{2}(2k^2+6k+3))}{(2k^2+4k+3)^2}$$

$$3 = \frac{(6r^2-s^2+i\sqrt{2}(6r^2+6rs+s^2))(6r^2-s^2-i\sqrt{2}(6r^2+6rs+s^2))}{(6r^2+4rs+s^2)^2}$$

The repetition of the above process exhibits four more distinct integer solutions to (1).

Way 2:

(3) is written as

$$u^2 + 2v^2 = 3w^6 \times 1 \tag{8}$$

Write 1 on the R.HLS. of (8) as

$$1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9} \tag{9}$$

Substituting (4), (5) and (9) in (8) and employing the method of factorization, define

$$u + i\sqrt{2}v = \frac{(1+i\sqrt{2})(1+i2\sqrt{2})(a+i\sqrt{2}b)^6}{3}$$

On equating the real and imaginary parts ,one obtains

$$u = -(a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6 - 2(6a^5b - 40a^3b^3 + 24ab^5), \quad (10)$$
$$v = a^6 —30a^4b^2 + 60a^2b^4 - 8b^6 - (6a^5b - 40a^3b^3 + 24ab^5)$$

From (4), (10) and (2), one has

$$x = 72(-2(a^6 - 30a^4b^2 + 60a^2b^4 - 8b^6) - (6a^5b - 40a^3b^3 + 24ab^5)),$$
$$y = -216(6a^5b - 40a^3b^3 + 24ab^5), \quad (7)$$
$$z = 2a^2 + 4b^2$$

which represent the non-zero distinct integer solutions to (1).

Way 3:

Write (3) in the form of ratio as

$$\frac{u+w^3}{w^3+v}=\frac{2(w^3-v)}{u-w^3}=\frac{\alpha}{\beta},\beta\neq 0 \tag{11}$$

which is equivalent to the system of double equations

$$\beta u - \alpha v + (\beta - \alpha)w^3 = 0$$
$$\alpha u + 2\beta v - (2\beta + \alpha)w^3 = 0$$

Applying the method of cross-multiplication, one has

$$u = \alpha^2 + 4\alpha\beta - 2\beta^2 \tag{12}$$
$$y = -\alpha^2 + 2\alpha\beta + 2\beta$$
$$w = \alpha^2 + 2\beta^2 \tag{13}$$

It is seen that (13) is satisfied by

$$\alpha = m(m^2 + 2n^2),\ \beta = n(m^2 + 2n^2) \tag{14}$$

and

$$w = m^2 + 2n^2 \tag{15}$$

Using (14) in (12), one has

$$u = (m^2 + 2n^2)^2\ (m^2 + 4mn - 2n^2)$$
$$v = (m^2 + 2n^2)^2\ (-m^2 + 2mn + 2n^2), \tag{16}$$

Substituting (15) and (16) in (2) ,the integer solutions to (1) are given by

$$x = 432mn(m^2 + 2n^2)^2,$$
$$y = 72(m^2 + 2n^2)^2\ (2m^2 + 2mn - 4n^2),$$
$$z = 2m^2 + 4n^2$$

Notes:

1. Using (4) in (13) and employing the method of factorization ,it is obtained that

$$\alpha = a - 6ab^2,\ \alpha = 3a^2b - 2b^2$$

In this case ,the integer solutions to (1) are given by

$$x = 432(a^2 - 6ab^2)(3a^2b - 2b^2),$$
$$y = 144[(a^2 - 6ab^2)^2 + (a^2 - 6ab^3)\ (3a^2b - 2b^2) - 2(3a^2b - 2b^2)^2],$$
$$z = 2a^2 + 4b^2$$

(3) is also written in the form of ratio as below:

$$\frac{u+w^2}{2(w^3+v)}=\frac{(w^3-v)}{u-w^3}=\frac{\alpha}{\beta},\beta\neq 0,$$

$$\frac{u + w^2}{2(w^3 - v)} = \frac{(w^3 + v)}{u - w^3} = \frac{\alpha}{\beta}, \beta \neq 0,$$

$$\frac{u + w^2}{(w^3 + v)} = \frac{2(w^3 + v)}{u - w^3} = \frac{\alpha}{\beta}, \beta \neq 0$$

The repetition of the above process exhibits three more distinct integer solutions to (1).

CONCLUSION

In this paper, an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous ternary sextic Diophantine equation given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the ternary sextic Diophantine equation under consideration.

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